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On Distinct Representatives and Mapping Theorems

PHILLIP A. OSTRAND

University of California at Santa Barbara, Santa Barbara, California

Submitted by Ky Fan

We present an elementary proof of a transfinite symmetric form of Philip Hall's theorem on systems of distinct representatives and derive some well-known mapping theorems from it to illustrate the ease with which this form of the theorem may be used.

NOTATION AND TERMINOLOGY

$|S|$ denotes the cardinality of the set S. $S \Subset T$ means that S is a finite subset of T. For $T \subset X \times Y$, $A \subset X$, $B \subset Y$ we shall use the functional notation

$$T(A) = \{y \in Y \mid (x, y) \in T \quad \text{for some } x \in A\},$$
$$T^{-1}(B) = \{x \in X \mid (x, y) \in T \quad \text{for some } y \in B\}.$$

For $T \subset X \times Y$, $U \subset X$, $V \subset Y$ we say that T *spans* U if

$$|T(A)| \geqslant |A| \qquad \text{for all } A \Subset U, \tag{1}$$

and T *spans* V if

$$|T^{-1}(B)| \geqslant |B| \qquad \text{for all } B \Subset V. \tag{2}$$

There is some ambiguity in the meaning of T spans U if U is a subset of both X and Y. This could be avoided by defining the *left spanning* of subsets of X and *right spanning* of subsets of Y by conditions (1) and (2) respectively. However, since no misunderstanding is likely to occur, we prefer the simplicity of the ambiguous terminology.

A subset T of $X \times Y$ is *scattered* if both T and T^{-1} are functions; i.e., $|T(x)| \leqslant 1$ for all $x \in X$ and $|T^{-1}(y)| \leqslant 1$ for all $y \in Y$. For $U \subset X$, $V \subset Y$ and $T \subset X \times Y$, $S(T, U, V)$ is the set of all subsets S of T which span both U and V and $M(T, U, V)$ is the set of members of $S(T, U, V)$ which are minimal under inclusion, i.e., which do not properly include a member of $S(T, U, V)$.

THEOREM 1. *Let $T \subset X \times Y$, $U \subset X$ and $V \subset Y$ such that $T(x)$ and $T^{-1}(y)$ are finite for each $x \in U$ and $y \in V$. There is a scattered set $S \subset T$ which spans U and V if and only if T spans U and V.*

For X and Y finite this theorem, expressed in terms of matrices, was proved by Mendelsohn and Dulmage [4]. For $U = X$ and $V = \varnothing$ this is the normal transfinite form of Philip Hall's theorem, which was first proved by Marshall Hall [3], in which the sets being represented are the $T(x)$, $x \in X$. The proof we offer is elementary and does not use any of the weaker forms of Hall's theorem.

It is obvious that spanning is a property of sets which is preserved under enlargement, so $S(T, U, V) \neq \varnothing$ if and only if T spans U and V. Thus one of the implications of the theorem is immediate. The other implication is proved by showing firstly that $S(T, U, V) \neq \varnothing$ implies $M(T, U, V) \neq \varnothing$, and secondly that $S \in M(T, U, V)$ implies that S is scattered.

LEMMA 1. *For $T \subset X \times Y$, $U \subset X$ and $V \subset Y$, if $T(x)$ and $T^{-1}(y)$ are finite for each $x \in U$ and $y \in V$ then $S(T, U, V) \neq \varnothing$ implies $M(T, U, V) \neq \varnothing$.*

Proof. Suppose $S(T, U, V) \neq \varnothing$. Let α be a chain in $S(T, U, V)$ ordered by inclusion. Let $M = \cap \{A \mid A \in \alpha\}$. Let $I \Subset U$. For $x \in I$ and $A \in \alpha$, $A(x) \subset T(x)$ which is finite. Thus for $x \in I$, $\{A(x) \mid A \in \alpha\}$ is a chain of finite sets. Hence there is an $A_x \in \alpha$ such that $A_x(x) \subset A(x)$ for all $A \in \alpha$ $\{A_x \mid x \in I\}$ is a finite chain, so it has a minimal member A_0. Thus $A_0(x) \subset A_x(x) \subset A(x)$ for all $x \in I$, $A \in \alpha$. It follows that $A_0(x) \subset M(x)$ for all $x \in I$. Consequently $A_0(I) \subset M(I)$, so $|M(I)| \geqslant |A_0(I)| \geqslant |I|$. A similar argument shows that $|M^{-1}(J)| \geqslant |J|$ for all $J \Subset V$. Thus $M \in S(T, U, V)$. The desired conclusion follows by Zorn's Lemma.

It remains to show that if $S \in M(T, U, V)$ then S is scattered. Toward that end we define additional terminology and deduce some preliminary technical results which we present in the form of remarks. For convenience of reference we state the several remarks together. Their proofs then follow. Each remark consists of two similar statements, inverses of each other in a natural sense, of which we prove only the first.

Let $S \in S(T, U, V)$. For $I \Subset U$, $|S(I)| \geqslant |I|$. If $|S(I)| = |I|$ we say that I is *critical*. A point x of U is *critical* if it is a member of a critical subset of U. Critical subsets and points of V are defined in the analogous manner.

REMARKS

Let $S \in S(T, U, V)$.

R1. Let $(x, y) \in S$. If x is not critical then $R = S - \{(x, y)\}$ spans U. Similarly if y is not critical then R spans V.

R2. If $H, I \Subset U$ are critical then $H \cap I$ and $H \cup I$ are also critical and $S(H-I) \cap S(I-H) \subset S(H \cap I)$. The analogous statement holds for critical subsets of V.

R3. If $x \in U$ is critical then there is a critical $I_x \Subset U$ which contains x and is contained in every other critical subset of U which contains x. In the analogous situation the minimal critical set containing $y \in V$ is denoted J_y.

R4. If $x \in U$ is critical then there is at most one $y \in S(x)$ such that $|I_x \cap S^{-1}(y)| = 1$. Similarly if $y \in V$ is critical there is at most one $x \in S^{-1}(y)$ such that $|J_y \cap S(x)| = 1$.

R5. Let $(x, y) \in S$. If x is critical and $|I_x \cap S^{-1}(y)| \geqslant 2$ then $R = S - \{(x, y)\}$ spans U. Similarly if y is critical and $|J_y \cap S(x)| \geqslant 2$ then R spans V.

Proofs

R1. Let $I \Subset U$. Either I is not critical, in which case $|S(I)| > |I|$ and hence $|R(I)| \geqslant |S(I)| - 1 \geqslant |I|$, or $x \notin I$, in which case $R(I) = S(I)$ so $|R(I)| = |S(I)| \geqslant |I|$.

R2. Let $A = S(H \cap I)$, $B = S(H-I) - A$ and $C = S(I-H) - A$. Then

$$\begin{aligned} |S(H \cap I)| + |S(H \cup I)| &\geqslant |H \cap I| + |H \cup I| \\ &= |H| + |I| \\ &= |A \cup B| + |A \cup C| \\ &= 2|A| + |B| + |C| \\ &\geqslant |A| + |A \cup B \cup C| \\ &= |S(H \cap I)| + |S(H \cup I)|. \end{aligned} \tag{3}$$

It follows that both inequalities in (3) are in fact equalities. In the first case this implies that $H \cap I$ and $H \cup I$ are critical. In the second case we see that B and C are disjoint and thus the desired conclusion follows.

R3. Let I_x be a critical subset of U of minimal cardinality among those containing x. If I is any critical set containing x then $I_x \cap I$ is critical by R2 and contains x. By the minimality, $|I_x \cap I| \geqslant |I_x|$, so $I_x \subset I$.

R4. Suppose $y, z \in S(x)$ and $|I_x \cap S^{-1}(y)| = |I_x \cap S^{-1}(z)| = 1$. Then $I_x \cap S^{-1}(y) = I_x \cap S^{-1}(z) = \{x\}$. Let $I = I_x - \{x\}$. Then $S(I) \subset S(I_x) - \{y, z\}$. But then $|S(I)| \leqslant |S(I_x)| - 2 = |I_x| - 2 < |I|$ contradicting the fact that S spans U.

R5. By hypothesis there is a $z \neq x$ in I_x with (z, y) in S and thus in R. Let $I \Subset U$. If I is not critical or $x \notin I$ then as in the proof of R1 we have $|R(I)| \geqslant |I|$. If I is critical and $x \in I$ then $I_x \subset I$ and thus $z \in I$. It follows that $y \in R(I)$ so $R(I) = S(I)$ and $|R(I)| = |S(I)| \geqslant |I|$.

Two lemmas now complete the proof. As with the preceding remarks each is in two parts, only one of which is proved. Together they show that minimal spanning sets are scattered.

LEMMA 2. *Let $(x, y) \in S \in M(T, U, V)$. If x is not critical then $|S(x)| = 1$ and similarly if y is not critical then $|S^{-1}(y)| = 1$.*

Proof. Suppose $|S(x)| > 1$. Then there is a $z \neq y$ in $S(x)$. Let $R = S - \{(x, y)\}$ and $R' = S - \{(x, z)\}$. R1 shows that R and R' span U. By the minimality of S we conclude that neither R nor R' spans V. Then by R1 again y and z must be critical and thus by R5,

$$|J_y \cap S(x)| = |J_z \cap S(x)| = 1.$$

Then $y \notin J_z$ and $z \notin J_y$. Let $J = J_y \cap J_z$. Then $y \in J_y - J_z$ and $z \in J_z - J_y$ so $x \in S^{-1}(J_y - J_z) \cap S^{-1}(J_z - J_y)$. Therefore, by R2 we have $x \in S^{-1}(J)$. Then there is a $w \in J$ with $(x, w) \in S$, so $|J_y \cap S(x)| \geqslant |\{w, y\}| = 2$ which implies by R5 that R spans V, contradicting the minimality of S.

LEMMA 3. *Let $(x, y) \in S \in M(T, U, V)$. If x is critical then $|S(x)| = 1$ and similarly if y is critical then $|S^{-1}(y)| = 1$.*

Proof. We proceed by induction on $|I_x|$. If $|I_x| = 1$ then $\{x\}$ is critical so $|S(x)| = |\{x\}| = 1$. Let $|I_x| > 1$ and suppose $|S(x)| > 1$. By R4 there exists a z in $S(x)$ for which $|I_x \cap S^{-1}(z)| \geqslant 2$. Then there is a $w \neq x$ in $I_x \cap S^{-1}(z)$. Let $R = S - \{(x, z)\}$ and $R' = S - \{(w, z)\}$. R5 shows that R spans U, so the minimality of S implies that R does not span V. Lemma 5 shows z must be critical since $|S^{-1}(z)| > 1$. Hence $|J_z \cap S(x)| = 1$ by R5. Then $|J_z \cap S(w)| \geqslant 2$ by R4 and hence R' spans V by R5. I_x is a critical set containing w, so w is critical and $I_w \subset I_x$. If $I_w \neq I_x$ then $|I_w| < |I_x|$ so inductively we have $|S(w)| = 1$ contradicting $|J_z \cap S(w)| \geqslant 2$. Thus $I_w = I_x$. But then $|I_w \cap S^{-1}(z)| = |I_x \cap S^{-1}(z)| \geqslant 2$ so R' spans U by R5, contradicting the minimality of S.

We shall now use Theorem 1 to derive several well-known mapping theorems. We feel that these derivations have a certain "slick" quality which illustrates the scope of Theorem 1 as a tool.

THEOREM 2 (Schroeder–Bernstein). *Let $\alpha : X \to Y$ and $\beta : Y \to X$ be injective mappings. Then there is a bijective mapping $\gamma : X \to Y$ which is contained in $\alpha \cup \beta^{-1}$.*

Proof. Let $T = \alpha \cup \beta^{-1} = \{(x, y) \in X \times Y \mid \alpha(x) = y \text{ or } \beta(y) = x\}$. Since α and β are injective we have

$$\begin{aligned} |T(x)| &\leqslant 2 && \text{for all } x \in X,\\ |T^{-1}(y)| &\leqslant 2 && \text{for all } y \in Y,\\ |T(I)| &\geqslant |\alpha(I)| = |I| && \text{for all } I \Subset X,\\ |T^{-1}(J)| &\geqslant |\beta(J)| = |J| && \text{for all } J \Subset Y. \end{aligned}$$

Thus by Theorem 1 there exists a scattered set $\gamma \subset T$ which spans X and Y. Then γ and γ^{-1} are functions so $\gamma : X \to Y$ is a bijection.

A more general result is

THEOREM 3 (O. Ore [5]). *Let $\bar{X} \subset X$ and $\bar{Y} \subset Y$ and let $\alpha : \bar{X} \to Y$ and $\beta : \bar{Y} \to X$ be injective mappings. Then there exist sets X^* and Y^* with $\bar{X} \subset X^* \subset X$ and $\bar{Y} \subset Y^* \subset Y$ and a bijective mapping $\gamma : X^* \to Y^*$ which is contained in $\alpha \cup \beta^{-1}$.*

The proof of this result proceeds identically with that of Theorem 2, beginning with $T = \alpha \cup \beta^{-1}$ and culminating with a scattered set $\gamma \subset T$ which spans $\bar{X}$ and $\bar{Y}$. Clearly γ is a bijection of subsets of X and Y the domain and range of which contain $\bar{X}$ and $\bar{Y}$ respectively.

Another classic theorem related to the preceding results is

THEOREM 4 (S. Banach [1]). *Let $\alpha : X \to Y$ and $\beta : Y \to X$ be injective mappings. Then there exist partitions $X = A_1 \cup B_1$ and $Y = A_2 \cup B_2$ such that $\alpha(A_1) = B_2$ and $\beta(A_2) = B_1$.*

Theorem 4 has been generalized in two directions. A proof is obtained by specializing that of Theorems 5 or 6.

THEOREM 5 (H. Perfect and J. S. Pym [6]). *Let $\bar{X} \subset X$ and $\bar{Y} \subset Y$ and let $\alpha : \bar{X} \to Y$ and $\beta : \bar{Y} \to X$ be injective mappings. Then there exist sets X^* and Y^* and partitions $X^* = A_1 \cup B_1$, $Y^* = A_2 \cup B_2$ with $A_1 \subset \bar{X} \subset X^* \subset X$ and $A_2 \subset \bar{Y} \subset Y^* \subset Y$ such that $\alpha(A_1) = B_2$ and $\beta(A_2) = B_1$.*

Proof. Let $T = \alpha \cup \beta^{-1}$. It is easily seen as in Theorem 2 that T satisfies the hypothesis of Theorem 1. Let S be a scattered subset of T which spans $\bar{X}$ and $\bar{Y}$. Let

$$\begin{aligned} A_1 &= \{x \in \bar{X} \mid (x, \alpha(x)) \in S\},\\ B_2 &= \alpha(A_1),\\ A_2 &= \bar{Y} - B_2,\\ B_1 &= \beta(A_2). \end{aligned}$$

We need only verify that A_1 and B_1 are disjoint and that $\bar{X}$ is contained in their union. Let $x \in B_1$. Then $x = \beta(y)$ for some $y \in A_2$. Then $y \in \bar{Y}$ so we have $(x', y) \in S$ for some $x' \in X$, and $y \notin B_2$ so we have $y \neq \alpha(x')$. Then $x' = \beta(y) = x$. Thus $(x, y) \in S$ and $y \neq \alpha(x)$ so $x \notin A_1$. Let $x \in \bar{X}$. Then $(x, y) \in S$ for some $y \in Y$. If $x \notin A_i$ then $y \neq \alpha(x)$. Then $x = \beta(y)$ so $y \in \bar{Y}$, and $y \notin B_2$. Then $y \in A_2$ so $x = \beta(y) \in B_1$.

THEOREM 6 (K. Fan [2]). *For $1 \leqslant i \leqslant 2n$ let $\varphi_i : X_i \to X_{i+1}$ be an injective mapping, where by X_{2n+1} we mean X_1. Then there exist partitions $X_i = A_i \cup B_i$ such that $\varphi_i(A_i) = B_{i+1}$.*

Proof. Let X be the disjoint union of the sets X_{2i-1} and Y the disjoint union of the sets X_{2i}, $1 \leqslant i \leqslant n$. Let

$$T = \left(\bigcup_{i=1}^{n} \varphi_{2i-1}\right) \cup \left(\bigcup_{i=1}^{n} \varphi_{2i}^{-1}\right) \subset X \times Y.$$

Again it is easily verified that T has the properties required by Theorem 1. Let S be a scattered subset of T which spans X and Y. Effect the desired partitions by taking

$$A_{2i-1} = \{x \in X_{2i-1} \mid (x, \varphi_{2i-1}(x)) \in S\} \qquad \text{and} \qquad B_{2i} = \varphi_{2i-1}(A_{2i-1}).$$

We are then finished if we show that $\varphi_{2i}(A_{2i}) = B_{2i+1}$. Let $y \in A_{2i}$. There is a unique $x \in X$ such that $(x, y) \in S$. Since $y \notin B_{2i}$ we see that $y \neq \varphi_{2i-1}(x)$. Hence $x = \varphi_{2i}(y)$, and $x \notin A_{2i+1}$. Thus $\varphi_{2i}(y) \in B_{2i+1}$. Conversely, let $x \in B_{2i+1}$. There is a unique $y \in Y$ such that $(x, y) \in S$. $x \notin A_{2i+1}$ implies that $y \neq \varphi_{2i+1}(x)$. Then $x = \varphi_{2i}(y)$ and $y \notin B_{2i}$. Thus $x \in \varphi_{2i}(A_{2i})$.

The ideas of Theorems 5 and 6 may be combined with the result,

THEOREM 7. *For $1 \leqslant i \leqslant 2n$ let $\bar{X}_i \subset X_i$ and let $\varphi_i : \bar{X}_i \to X_{i+1}$ be an injective mapping, where by X_{2n+1} and $\bar{X}_{2n+1}$ we mean X_1 and $\bar{X}_1$ respectively. Then there exist sets X_i^* and partitions $X_i^* = A_i \cup B_i$ with $A_i \subset \bar{X}_i \subset X_i^* \subset X_i$ such that $\varphi_i(A_i) = B_{i+1}$ for $1 \leqslant i \leqslant 2n$.*

A proof of this theorem is easily concocted by combining the technique used in proving Theorems 5 and 6.

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